Chain Rule Example

Let $z = f(x,y)$.

Let $x = r\cos\theta$.

Let $y = r\sin\theta$.

Find $\frac{\partial^2 z}{\partial \theta^2}$ in terms of x and y.

Soln:

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}\cdot(r\cos\theta)\right)$$

Product Rule $\longrightarrow$

$$= \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right)(-r\sin\theta) + \frac{\partial}{\partial \theta}(-r\sin\theta)\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)(r\cos\theta) + \frac{\partial}{\partial \theta}(r\cos\theta)\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial y}{\partial \theta}\right)\right)(-r\sin\theta) + \left(\frac{\partial z}{\partial x}\right)(-r\cos\theta) + \left(\frac{\partial^2 z}{\partial y \partial x}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial \theta}\right)\right)(r\cos\theta) + \left(\frac{\partial z}{\partial y}\right)(-r\sin\theta)$$

$$= \left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial x \partial y}(r\cos\theta)\right)(-r\sin\theta) + \left(\frac{\partial z}{\partial x}\right)(-x) + \left(\frac{\partial^2 z}{\partial y \partial x}(-r\sin\theta) + \frac{\partial^2 z}{\partial y^2}(r\cos\theta)\right)(r\cos\theta) +$$

$$\left(\frac{\partial z}{\partial y}\right)(-y)$$

$$=(r^2\sin^2\theta)\left(\frac{\partial^2 z}{\partial x^2}\right)-(r^2\sin\theta\cos\theta)\left(\frac{\partial^2 z}{\partial x\partial y}\right)-(x)\left(\frac{\partial z}{\partial x}\right)$$

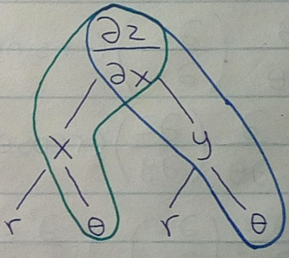
$$-(r^2\sin\theta\cos\theta)\left(\frac{\partial^2 z}{\partial y\partial x}\right)+(r^2\cos^2\theta)\left(\frac{\partial^2 z}{\partial y^2}\right)-(y)\left(\frac{\partial z}{\partial y}\right)$$

$$=\left(\frac{\partial^2 z}{\partial x^2}\right)(y^2)-(2xy)\left(\frac{\partial^2 z}{\partial x\partial y}\right)+(x^2)\left(\frac{\partial^2 z}{\partial y^2}\right)-(x)\left(\frac{\partial z}{\partial x}\right)$$

$$-(y)\left(\frac{\partial z}{\partial y}\right)$$

Note:

1. $\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial\theta}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial\theta}$



2. $\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial\theta}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial\theta}$